

THE SECOND HOCHSCHILD COHOMOLOGY GROUP FOR A CLASS OF ONE-PARAMETRIC SELF-INJECTIVE ALGEBRAS

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ABSTRACT. In this paper we determine the second Hochschild cohomology group for a class of self-injective algebras of tame representation type namely, those which are standard one-parametric but not weakly symmetric. These were classified up to derived equivalence by Bocian, Holm and Skowroński in [2]. We show for the algebras $\Lambda = \Lambda(p, q, k, s, \lambda)$ that $\mathrm{HH}^2(\Lambda)$ has dimension 1 and we find an associative deformation of Λ .

INTRODUCTION

This paper determines the second Hochschild cohomology group for all standard one-parametric but not weakly symmetric self-injective algebras of tame representation type. Bocian, Holm and Skowroński give, in [2], a classification of these algebras by quiver and relations up to derived equivalence. The algebras in [2] are divided into two types and here we study the second Hochschild cohomology group for one type, namely for the algebras $\Lambda = \Lambda(p, q, k, s, \lambda)$ where p, q, s, k are integers such that $p, q \geq 0, k \geq 2, 1 \leq s \leq k-1, \gcd(s, k) = 1, \gcd(s+2, k) = 1$ and $\lambda \in K \setminus \{0\}$.

We start, in Section 1, by introducing the algebra Λ by quiver and relations. Section 2 of this paper describes the projective resolution of [3] which we use to find $\mathrm{HH}^2(\Lambda)$. In the third section, we determine $\mathrm{HH}^2(\Lambda)$ explicitly, considering separately the cases $1 \leq s \leq k-2$ and $s = k-1$. The main result is Theorem 3.9, which shows that $\mathrm{HH}^2(\Lambda)$ has dimension 1 for $1 \leq s \leq k-1$. This is in contrast to the majority of self-injective algebras of finite representation type (see [1]). Since Hochschild cohomology is invariant under derived equivalence, the second Hochschild cohomology group is now known for all standard one-parametric but not weakly symmetric self-injective algebras of tame representation type. We conclude the section with Theorem 3.10 where we find a non-trivial deformation Λ_η of Λ associated to our non-zero element η in $\mathrm{HH}^2(\Lambda)$. This illustrates the connection between the second Hochschild cohomology group and deformation theory.

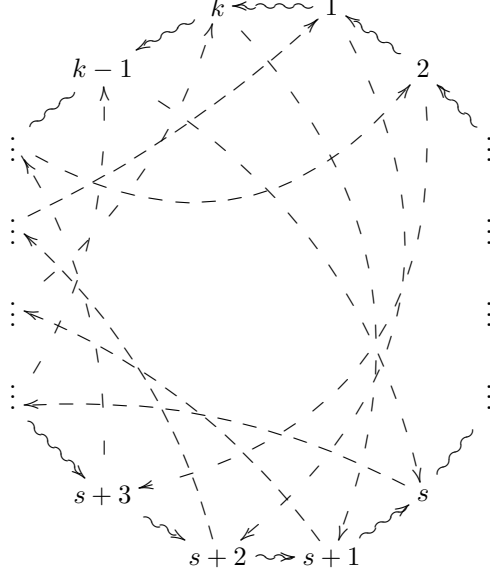
1. A CLASS OF ONE-PARAMETRIC SELF-INJECTIVE ALGEBRAS

We start by describing the algebras $\Lambda = \Lambda(p, q, k, s, \lambda)$ of [2]. Let K be an algebraically closed field and let p, q, s, k be integers such that $p, q \geq 0, k \geq 2, 1 \leq s \leq k-1, \gcd(s, k) = 1, \gcd(s+2, k) = 1$ and $\lambda \in K \setminus \{0\}$. From [2, Section 5], $\Lambda(p, q, k, s, \lambda)$ has quiver $\mathcal{Q}(p, q, k, s)$:

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where, for any $i \in \{1, 2, \dots, k\}$, $i \rightsquigarrow i-1$ denotes the path

$$i \xrightarrow{\alpha_{(i,0)}} (i, 1) \xrightarrow{\alpha_{(i,1)}} (i, 2) \xrightarrow{\alpha_{(i,2)}} \dots \xrightarrow{\alpha_{(i,q-1)}} (i, q) \xrightarrow{\alpha_{(i,q)}} i-1,$$

and $i-1 \dashrightarrow i+s$ denotes the path

$$i-1 \xrightarrow{\beta_{i0}} i^1 \xrightarrow{\beta_{i1}} i^2 \xrightarrow{\beta_{i2}} \dots \xrightarrow{\beta_{ip-1}} i^p \xrightarrow{\beta_{ip}} i+s.$$

Then $\Lambda = K\mathcal{Q}(p, q, k, s)/I(p, q, k, s, \lambda)$ where $I(p, q, k, s, \lambda)$ is the ideal generated by the relations

- $\beta_{ip}\beta_{(s+i+1)^0}$, for $i = 1, 2, \dots, k$,
- $\alpha_{(i,q)}\alpha_{(i-1,0)}$, for $i = 1, 2, \dots, k$,
- $\alpha_{(i,t')}\alpha_{(i,t'+1)} \dots \alpha_{(i,q)}\beta_{i^0}\beta_{i^1} \dots \beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)} \dots \alpha_{(s+i,t')}$,
for $t' = 0, 1, \dots, q$, $i = 1, 2, \dots, k$,
- $\beta_{ij}\beta_{ij+1} \dots \beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)} \dots \alpha_{(s+i,q)}\beta_{(s+i)^0}\beta_{(s+i)^1} \dots \beta_{(s+i)^j}$,
for $j = 0, 1, \dots, p$, $i = 1, 2, \dots, k$,
- $\alpha_{(i,0)}\alpha_{(i,1)} \dots \alpha_{(i,q)}\beta_{i^0}\beta_{i^1} \dots \beta_{i^p}$
 $-\beta_{(i+1)^0}\beta_{(i+1)^1} \dots \beta_{(i+1)^p}\alpha_{(s+i+1,0)}\alpha_{(s+i+1,1)} \dots \alpha_{(s+i+1,q)}$,
for $i = 2, \dots, k$, and
- $\alpha_{(1,0)}\alpha_{(1,1)} \dots \alpha_{(1,q)}\beta_{1^0}\beta_{1^1} \dots \beta_{1^p} - \lambda\beta_{2^0}\beta_{2^1} \dots \beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)} \dots \alpha_{(s+2,q)}$,
where $\lambda \in K \setminus \{0\}$.

Note that we write our paths from left to right.

In order to compute $\text{HH}^2(\Lambda)$ for $\Lambda = \Lambda(p, q, k, s, \lambda)$, the next section gives the necessary background required to find the first terms of the projective resolution of Λ as a Λ, Λ -bimodule. Section 3 uses this part of a minimal projective bimodule

resolution for our algebras to determine the second Hochschild cohomology group and provides the main results of this paper.

2. PROJECTIVE RESOLUTIONS

To find the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \Lambda(p, q, k, s, \lambda)$, we use the projective resolution of [3]. More generally, let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra, where K is an algebraically closed field, \mathcal{Q} is a quiver, and I is an admissible ideal of $K\mathcal{Q}$. Fix a minimal set f^2 of generators for the ideal I . Let $x \in f^2$. Then $x = \sum_{j=1}^r c_j a_{1j} \cdots a_{kj} \cdots a_{s_j j}$, that is, x is a linear combination of paths $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ for $j = 1, \dots, r$ and $c_j \in K$ and there are unique vertices v and w such that each path $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ starts at v and ends at w for all j . We write $\mathfrak{o}(x) = v$ and $\mathfrak{t}(x) = w$. Similarly $\mathfrak{o}(a)$ is the origin of the arrow a and $\mathfrak{t}(a)$ is the end of a .

In [3, Theorem 2.9], it is shown that there is a minimal projective resolution of Λ as a Λ, Λ -bimodule which begins:

$$\cdots \rightarrow Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \rightarrow 0,$$

where the projective Λ, Λ -bimodules Q^0, Q^1, Q^2 are given by

$$Q^0 = \bigoplus_{v, \text{vertex}} \Lambda v \otimes v \Lambda,$$

$$Q^1 = \bigoplus_{a, \text{arrow}} \Lambda \mathfrak{o}(a) \otimes \mathfrak{t}(a) \Lambda, \text{ and}$$

$$Q^2 = \bigoplus_{x \in f^2} \Lambda \mathfrak{o}(x) \otimes \mathfrak{t}(x) \Lambda,$$

and the maps g, A_1, A_2 and A_3 are Λ, Λ -bimodule homomorphisms, defined as follows. The map $g : Q^0 \rightarrow \Lambda$ is the multiplication map so is given by $v \otimes v \mapsto v$. The map $A_1 : Q^1 \rightarrow Q^0$ is given by $\mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes \mathfrak{o}(a)a - a\mathfrak{t}(a) \otimes \mathfrak{t}(a)$ for each arrow a . With the notation for $x \in f^2$ given above, the map $A_2 : Q^2 \rightarrow Q^1$ is given by $\mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^r c_j (\sum_{k=1}^{s_j} a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j})$, where $a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j} \in \Lambda \mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj}) \Lambda$.

In order to describe the projective bimodule Q^3 and the map A_3 in the Λ, Λ -bimodule resolution of Λ in [3], we need to introduce some notation from [4]. Recall that an element $y \in K\mathcal{Q}$ is uniform if there are vertices v, w such that $y = vy = yw$. We write $\mathfrak{o}(y) = v$ and $\mathfrak{t}(y) = w$. In [4], Green, Solberg and Zacharia show that there are sets f^n in $K\mathcal{Q}$, for $n \geq 3$, consisting of uniform elements $y \in f^n$ such that $y = \sum_{x \in f^{n-1}} xrx = \sum_{z \in f^{n-2}} zsz$ for unique elements $r_x, s_z \in K\mathcal{Q}$ such that $s_z \in I$. These sets have special properties related to a minimal projective Λ -resolution of Λ/\mathfrak{r} , where \mathfrak{r} is the Jacobson radical of Λ . Specifically the n -th projective in the minimal projective Λ -resolution of Λ/\mathfrak{r} is $\bigoplus_{y \in f^n} \mathfrak{t}(y) \Lambda$.

In particular, to determine the set f^3 , we follow explicitly the construction given in [4, §1]. Let f^1 denote the set of arrows of \mathcal{Q} . Consider the intersection $(\bigoplus_i f_i^2 K\mathcal{Q}) \cap (\bigoplus_j f_j^1 I)$. Set this intersection equal to some $(\bigoplus_l f_l^{3*} K\mathcal{Q})$. We then discard all elements of the form f^{3*} that are in $\bigoplus_i f_i^2 I$; the remaining ones form precisely the set f^3 .

Thus, for $y \in f^3$ we have that $y \in (\bigoplus_i f_i^2 K\mathcal{Q}) \cap (\bigoplus_j f_j^1 I)$. So we may write $y = \sum f_i^2 p_i = \sum q_i f_i^2 r_i$ with $p_i, q_i, r_i \in K\mathcal{Q}$, such that p_i, q_i are in the ideal generated by the arrows of $K\mathcal{Q}$, and p_i unique. Then [3] gives that $Q^3 = \bigoplus_{y \in f^3} \Lambda \mathfrak{o}(y) \otimes \mathfrak{t}(y) \Lambda$

and, for $y \in f^3$ in the notation above, the component of $A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y))$ in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 is $\mathfrak{o}(y) \otimes p_i - q_i \otimes r_i$.

Applying $\text{Hom}(-, \Lambda)$ to this part of a minimal projective bimodule resolution of Λ gives us the complex

$$0 \rightarrow \text{Hom}(Q^0, \Lambda) \xrightarrow{d_1} \text{Hom}(Q^1, \Lambda) \xrightarrow{d_2} \text{Hom}(Q^2, \Lambda) \xrightarrow{d_3} \text{Hom}(Q^3, \Lambda)$$

where d_i is the map induced from A_i for $i = 1, 2, 3$. Then $\text{HH}^2(\Lambda) = \text{Ker } d_3 / \text{Im } d_2$.

Throughout, all tensor products are tensor products over K , and we write \otimes for \otimes_K . When considering an element of the projective Λ, Λ -bimodule $Q^1 = \bigoplus_{a, \text{arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ it is important to keep track of the individual summands of Q^1 . So to avoid confusion we usually denote an element in the summand $\Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ by $\lambda \otimes_a \lambda'$ using the subscript 'a' to remind us in which summand this element lies. Similarly, an element $\lambda \otimes_{f_i^2} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 and an element $\lambda \otimes_{f_i^3} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^3) \otimes \mathfrak{t}(f_i^3)\Lambda$ of Q^3 . We keep this notation for the rest of the paper.

3. $\text{HH}^2(\Lambda)$

We have given $\Lambda = \Lambda(p, q, k, s, \lambda)$ by quiver and relations in Section 1. However, these relations are not minimal. So next we will find a minimal set of relations f^2 for this algebra.

Let

$$\begin{aligned} f_{1,1}^2 &= \alpha_{(1,0)}\alpha_{(1,1)} \cdots \alpha_{(1,q)}\beta_{1^0}\beta_{1^1} \cdots \beta_{1^p} \\ &\quad - \lambda\beta_{2^0}\beta_{2^1} \cdots \beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)} \cdots \alpha_{(s+2,q)}, \\ f_{1,i}^2 &= \alpha_{(i,0)}\alpha_{(i,1)} \cdots \alpha_{(i,q)}\beta_{i^0}\beta_{i^1} \cdots \beta_{i^p} \\ &\quad - \beta_{(i+1)^0}\beta_{(i+1)^1} \cdots \beta_{(i+1)^p}\alpha_{(s+i+1,0)}\alpha_{(s+i+1,1)} \cdots \alpha_{(s+i+1,q)} \\ &\quad \text{for } i \in \{2, \dots, k\}, \\ f_{2,i}^2 &= \beta_{i^p}\beta_{(s+i+1)^0} \quad \text{for } i \in \{1, \dots, k\}, \\ f_{3,i}^2 &= \alpha_{(i,q)}\alpha_{(i-1,0)} \quad \text{for } i \in \{1, \dots, k\}, \\ f_{4,i,j}^2 &= \beta_{i^j}\beta_{i^{j+1}} \cdots \beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)} \cdots \alpha_{(s+i,q)}\beta_{(s+i)^0}\beta_{(s+i)^1} \cdots \beta_{(s+i)^j} \\ &\quad \text{where } j \in \{1, \dots, p-1\} \text{ and } i \in \{1, \dots, k\}, \\ f_{5,i,t'}^2 &= \alpha_{(i,t')}\alpha_{(i,t'+1)} \cdots \alpha_{(i,q)}\beta_{i^0}\beta_{i^1} \cdots \beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)} \cdots \alpha_{(s+i,t')} \\ &\quad \text{where } t' \in \{1, \dots, q-1\} \text{ and } i \in \{1, \dots, k\}. \end{aligned}$$

The remaining relations given in Section 1 are all linear combinations of the above relations. For example, the relation $\beta_{i^0}\beta_{i^1} \cdots \beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)} \cdots \alpha_{(s+i,q)}\beta_{(s+i)^0}$ can be written as $\alpha_{(i-1,0)}\alpha_{(i-1,1)} \cdots \alpha_{(i-1,q)}\beta_{(i-1)^0}\beta_{(i-1)^1} \cdots \beta_{(i-1)^{p-1}}f_{2,i-1}^2 - f_{1,i-1}^2\beta_{(s+i)^0}$. So this relation is in I and is not in f^2 .

Proposition 3.1. *For $\Lambda = \Lambda(p, q, k, s, \lambda)$ and with the above notation, the minimal set of relations is*

$$f^2 = \{f_{1,i}^2, f_{2,i}^2, f_{3,i}^2, f_{4,i,j}^2, f_{5,i,t'}^2\}.$$

In contrast to the majority of self-injective algebras of finite representation type, we will show that the algebra $\Lambda(p, q, k, s, \lambda)$ has non-zero second Hochschild cohomology group (see [1, Theorem 6.5]). Recall that $\text{HH}^2(\Lambda) = \text{Ker } d_3 / \text{Im } d_2$, where $d_3 : \text{Hom}(Q^2, \Lambda) \rightarrow \text{Hom}(Q^3, \Lambda)$ is induced by $A_3 : Q^3 \rightarrow Q^2$.

First we will find $\text{Im } d_2$. Since $d_2 : \text{Hom}(Q^1, \Lambda) \rightarrow \text{Hom}(Q^2, \Lambda)$, let $f \in \text{Hom}(Q^1, \Lambda)$ so that $d_2 f = f A_2$. We consider the cases $1 \leq s \leq k-2$ and $s = k-1$ separately.

Let $1 \leq s \leq k-2$ and

$$\begin{aligned} f(e_i \otimes_{\beta_{(i+1)^0}} e_{(i+1)^1}) &= c_{1,i} \beta_{(i+1)^0}, \\ f(e_{(i+1)^j} \otimes_{\beta_{(i+1)^j}} e_{(i+1)^{j+1}}) &= c_{2,i+1,j} \beta_{(i+1)^j} \text{ for } j \in \{1, \dots, p-1\}, \\ f(e_{(i+1)^p} \otimes_{\beta_{(i+1)^p}} e_{s+i+1}) &= c_{2,i+1,p} \beta_{(i+1)^p}, \\ f(e_i \otimes_{\alpha_{(i,0)}} e_{(i,1)}) &= c_{3,i} \alpha_{(i,0)}, \\ f(e_{(i,t')} \otimes_{\alpha_{(i,t')}} e_{(i,t'+1)}) &= c_{4,i,t'} \alpha_{(i,t')} \text{ for } t' \in \{1, \dots, q-1\} \text{ and} \\ f(e_{(i,q)} \otimes_{\alpha_{(i,q)}} e_{i-1}) &= c_{4,i,q} \alpha_{(i,q)}, \end{aligned}$$

where all coefficients $c_{1,i}, c_{2,i+1,j}$ for $j \in \{1, \dots, p-1\}$, $c_{2,i+1,p}, c_{3,i}, c_{4,i,t'}$ for $t' \in \{1, \dots, q-1\}$, $c_{4,i,q} \in K$. Now we find $f A_2$.

First we have, $f A_2(e_1 \otimes_{f_{1,1}^2} e_{s+1})$

$$\begin{aligned} &= f(e_1 \otimes_{\alpha_{(1,0)}} e_{(1,1)}) \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} \\ &\quad + \alpha_{(1,0)} f(e_{(1,1)} \otimes_{\alpha_{(1,1)}} e_{(1,2)}) \alpha_{(1,2)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} \\ &\quad + \cdots + \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q-1)} f(e_{(1,q)} \otimes_{\alpha_{(1,q)}} e_k) \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} \\ &\quad + \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} f(e_k \otimes_{\beta_{1^0}} e_{1^1}) \beta_{1^1} \cdots \beta_{1^p} \\ &\quad + \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} f(e_{1^1} \otimes_{\beta_{1^1}} e_{1^2}) \beta_{1^2} \cdots \beta_{1^p} \\ &\quad + \cdots + \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^{p-1}} f(e_{1^p} \otimes_{\beta_{1^p}} e_{s+1}) \\ &\quad - \lambda [f(e_1 \otimes_{\beta_{2^0}} e_{2^1}) \beta_{2^1} \cdots \beta_{2^p} \alpha_{(s+2,0)} \alpha_{(s+2,1)} \cdots \alpha_{(s+2,q)} \\ &\quad + \beta_{2^0} f(e_{2^1} \otimes_{\beta_{2^1}} e_{2^2}) \beta_{2^2} \cdots \beta_{2^p} \alpha_{(s+2,0)} \alpha_{(s+2,1)} \cdots \alpha_{(s+2,q)} \\ &\quad + \cdots + \beta_{2^0} \beta_{2^1} \cdots \beta_{2^{p-1}} f(e_{2^p} \otimes_{\beta_{2^p}} e_{s+2}) \alpha_{(s+2,0)} \alpha_{(s+2,1)} \cdots \alpha_{(s+2,q)} \\ &\quad + \beta_{2^0} \beta_{2^1} \cdots \beta_{2^p} f(e_{s+2} \otimes_{\alpha_{(s+2,0)}} e_{(s+2,1)}) \alpha_{(s+2,1)} \cdots \alpha_{(s+2,q)} \\ &\quad + \beta_{2^0} \beta_{2^1} \cdots \beta_{2^p} \alpha_{(s+2,0)} f(e_{(s+2,1)} \otimes_{\alpha_{(s+2,1)}} e_{(s+2,2)}) \alpha_{(s+2,2)} \cdots \alpha_{(s+2,q)} \\ &\quad + \cdots + \beta_{2^0} \beta_{2^1} \cdots \beta_{2^p} \alpha_{(s+2,0)} \alpha_{(s+2,1)} \cdots \alpha_{(s+2,q-1)} f(e_{(s+2,q)} \otimes_{\alpha_{(s+2,q)}} e_{s+1})] \\ &= (c_{3,1} + c_{4,1,1} + \cdots + c_{4,1,q} + c_{1,k} + c_{2,1,1} + \cdots + c_{2,1,p}) \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} \\ &\quad - \lambda (c_{1,1} + c_{2,2,1} + \cdots + c_{2,2,p} + c_{3,s+2} + c_{4,s+2,1} + \cdots + c_{4,s+2,q}) \beta_{2^0} \beta_{2^1} \cdots \beta_{2^p} \\ &\quad \alpha_{(s+2,0)} \alpha_{(s+2,1)} \cdots \alpha_{(s+2,q)} \\ &= (c_{3,1} + c_{4,1,1} + \cdots + c_{4,1,q} + c_{1,k} + c_{2,1,1} + \cdots + c_{2,1,p} - c_{1,1} - c_{2,2,1} - \cdots - \\ &\quad c_{2,2,p} - c_{3,s+2} - c_{4,s+2,1} - \cdots - c_{4,s+2,q}) \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p}. \end{aligned}$$

Similarly for $i \in \{2, \dots, k\}$, $f A_2(e_i \otimes_{f_{1,i}^2} e_{s+i}) = (c_{3,i} + c_{4,i,1} + \cdots + c_{4,i,q} + c_{1,i-1} + c_{2,i,1} + \cdots + c_{2,i,p} - c_{1,i} - c_{2,i+1,1} - \cdots - c_{2,i+1,p} - c_{3,s+i+1} - c_{4,s+i+1,1} - \cdots - c_{4,s+i+1,q}) \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p}$.

For the remaining terms, $f A_2(\mathfrak{o}(x) \otimes_x \mathfrak{t}(x)) = 0$ where $x \in \{f_{2,i}^2, f_{3,i}^2, f_{4,i,j}^2, f_{5,i,t'}^2\}$ for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, p-1\}$ and $t' \in \{1, \dots, q-1\}$.

Let $c'_i = c_{3,i} + c_{4,i,1} + \cdots + c_{4,i,q} + c_{1,i-1} + c_{2,i,1} + \cdots + c_{2,i,p} - c_{1,i} - c_{2,i+1,1} - \cdots - c_{2,i+1,p} - c_{3,s+i+1} - c_{4,s+i+1,1} - \cdots - c_{4,s+i+1,q}$ for $i = 1, \dots, k$

and $\rho_i = \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p}$ for $i = 1, \dots, k$.

Thus for $i \in \{1, \dots, k\}$ and $1 \leq s \leq k-2$, $f A_2$ is given by

$$\begin{aligned} f A_2(e_i \otimes_{f_{1,i}^2} e_{s+i}) &= c'_i \rho_i, \\ f A_2(e_{i^p} \otimes_{f_{2,i}^2} e_{(s+i+1)^1}) &= 0, \\ f A_2(e_{(i,q)} \otimes_{f_{3,i}^2} e_{(i-1,1)}) &= 0, \end{aligned}$$

$fA_2(e_{ij} \otimes_{f_{4,i,j}^2} e_{(s+i)^{j+1}}) = 0$ where $j \in \{1, \dots, p-1\}$ and
 $fA_2(e_{(i,t')} \otimes_{f_{5,i,t'}^2} e_{(s+i,t'+1)}) = 0$ where $t' \in \{1, \dots, q-1\}$,
 where $c'_1, \dots, c'_k \in K$ with $\sum_{i=1}^k c'_i = 0$. So $\dim \text{Im } d_2 = k-1$.

For $s = k-1$, we let

$$\begin{aligned} f(e_i \otimes_{\beta_{(i+1)^0}} e_{(i+1)^1}) &= c_{1,i} \beta_{(i+1)^0}, \\ f(e_{(i+1)^j} \otimes_{\beta_{(i+1)^j}} e_{(i+1)^{j+1}}) &= c_{2,i+1,j} \beta_{(i+1)^j} \text{ for } j \in \{1, \dots, p-1\}, \\ f(e_{(i+1)^p} \otimes_{\beta_{(i+1)^p}} e_i) &= c_{2,i+1,p} \beta_{(i+1)^p}, \\ f(e_i \otimes_{\alpha_{(i,0)}} e_{(i,1)}) &= c_{3,i} \alpha_{(i,0)}, \\ f(e_{(i,t')} \otimes_{\alpha_{(i,t')}} e_{(i,t'+1)}) &= c_{4,i,t'} \alpha_{(i,t')} \text{ for } t' \in \{1, \dots, q-1\} \text{ and} \\ f(e_{(i,q)} \otimes_{\alpha_{(i,q)}} e_{i-1}) &= c_{4,i,q} \alpha_{(i,q)} + d_{1,i} \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p}, \end{aligned}$$

where for all $i \in \{1, \dots, k\}$ the coefficients $c_{1,i}, c_{2,i+1,j}$ for $j \in \{1, \dots, p-1\}$, $c_{2,i+1,p}$, $c_{3,i}, c_{4,i,t'}$ for $t' \in \{1, \dots, q-1\}$, $c_{4,i,q}$, $d_{1,i}$ are in K .

Then we can find fA_2 for $i \in \{1, \dots, k\}$ in the same way as the previous case to see that it is given by

$$\begin{aligned} fA_2(e_i \otimes_{f_{1,i}^2} e_{i-1}) &= c'_i \rho_i \text{ where } c'_i, \rho_i \text{ as above,} \\ fA_2(e_{i^p} \otimes_{f_{2,i}^2} e_{i^1}) &= 0, \\ fA_2(e_{(i,q)} \otimes_{f_{3,i}^2} e_{(i-1,1)}) &= d_{1,i} \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p} \alpha_{(i-1,0)}, \\ fA_2(e_{ij} \otimes_{f_{4,i,j}^2} e_{(i-1)^{j+1}}) &= 0 \text{ where } j \in \{1, \dots, p-1\} \text{ and} \\ fA_2(e_{(i,t')} \otimes_{f_{5,i,t'}^2} e_{(i-1,t'+1)}) &= 0 \text{ where } t' \in \{1, \dots, q-1\}, \end{aligned}$$

where $c'_1, \dots, c'_k, d_{1,1}, \dots, d_{1,k} \in K$ with $\sum_{i=1}^k c'_i = 0$. Note that there is no dependency between the $d_{1,i}$. So $\dim \text{Im } d_2 = 2k-1$.

Proposition 3.2. *If $1 \leq s \leq k-2$, we have $\dim \text{Im } d_2 = k-1$. If $s = k-1$, we have $\dim \text{Im } d_2 = 2k-1$.*

Next we find $\text{Hom}(Q^2, \Lambda)$ and again consider the two cases separately. Let $1 \leq s \leq k-2$ and $h \in \text{Hom}(Q^2, \Lambda)$. Then h is defined by

$$\begin{aligned} \mathfrak{o}(f_{1,i}^2) \otimes \mathfrak{t}(f_{1,i}^2) &\mapsto d_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p} \text{ for } i \in \{1, 2, \dots, k\}, \\ \text{else} &\mapsto 0, \end{aligned}$$

where $d_i \in K$.

Therefore $\dim \text{Hom}(Q^2, \Lambda) = k$. Hence, $\dim \text{Ker } d_3 \leq k$.

For $s = k-1$ and $i \in \{1, 2, \dots, k\}$, h is given by

$$\begin{aligned} \mathfrak{o}(f_{1,i}^2) \otimes \mathfrak{t}(f_{1,i}^2) &\mapsto d_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p} + d'_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)}, \\ \mathfrak{o}(f_{3,i}^2) \otimes \mathfrak{t}(f_{3,i}^2) &\mapsto d''_i \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p} \alpha_{(i-1,0)}, \\ \text{else} &\mapsto 0, \end{aligned}$$

where d_i, d'_i, d''_i are in K for $i \in \{1, \dots, k\}$. Thus $\dim \text{Hom}(Q^2, \Lambda) = 3k$.

Proposition 3.3. *If $1 \leq s \leq k-2$, we have $\dim \text{Hom}(Q^2, \Lambda) = k$. If $s = k-1$, $\dim \text{Hom}(Q^2, \Lambda) = 3k$.*

Corollary 3.4. *If $1 \leq s \leq k-2$, we have $\dim \text{Ker } d_3 \leq k$. If $s = k-1$, $\dim \text{Ker } d_3 \leq 3k$.*

In order to find $\text{Ker } d_3$ and hence determine $\text{HH}^2(\Lambda)$ we start by giving a non-zero element in $\text{HH}^2(\Lambda)$ for all s .

Proposition 3.5. Define $h_1 \in \text{Hom}(Q^2, \Lambda)$ by

$$\begin{aligned} \mathfrak{o}(f_{1,1}^2) \otimes \mathfrak{t}(f_{1,1}^2) = e_1 \otimes e_{s+1} &\mapsto \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} = \rho_1, \\ \text{else} &\mapsto 0. \end{aligned}$$

Then h_1 is in $\text{Ker } d_3$.

Proof. We note that $\rho_1 \neq 0$ so h_1 is a non-zero map. To show that $h_1 \in \text{Ker } d_3$ we show that $h_1 A_3 = 0$. First, observe that $\rho_1 \beta_{(s+2)^0} = 0$ and $\rho_1 \alpha_{(s+1,0)} = 0$. Hence $\rho_1 \mathfrak{r} = 0$. Similarly we have $\mathfrak{r} \rho_1 = 0$.

Recall that $Q^3 = \coprod_{y \in f^3} \Lambda \mathfrak{o}(y) \otimes \mathfrak{t}(y) \Lambda$ where $y = \sum_u f_u^2 p_u = \sum_u q_u f_u^2 r_u$ and p_u, q_u are in the ideal generated by the arrows. For $y \in f^3$ the component of $A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y))$ in $\Lambda \mathfrak{o}(f_u^2) \otimes \mathfrak{t}(f_u^2) \Lambda$ is

$$\Sigma(\mathfrak{o}(y) \otimes_{f_u^2} p_u - q_u \otimes_{f_u^2} r_u).$$

Then

$$h_1 A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y)) = \Sigma_u (h_1(\mathfrak{o}(y) \otimes_{f_u^2} p_u) - q_u h_1(\mathfrak{o}(f_u^2) \otimes_{f_u^2} \mathfrak{t}(f_u^2) r_u)).$$

$$\text{Thus } h_1(\mathfrak{o}(y) \otimes_{f_u^2} p_u) = \begin{cases} \rho_1 p_u & \text{if } f_u^2 = f_{1,1}^2 \\ 0 & \text{otherwise.} \end{cases}$$

As p_u is in the arrow ideal of KQ , $\rho_1 p_u \in \rho_1 \mathfrak{r} = 0$. So we have $h_1(\mathfrak{o}(y) \otimes p_u) = 0$. Similarly $h_1(q_u \otimes_{f_u^2} r_u) = 0$ as $q_u \rho_1 r_u \in \mathfrak{r} \rho_1 r_u = 0$. Therefore $h_1 A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y)) = 0$ for all $y \in f^3$ so $h_1 A_3 = 0$. Thus $h_1 \in \text{Ker } d_3$ as required. \square

Theorem 3.6. For $\Lambda = \Lambda(p, q, k, s, \lambda)$ where p, q are positive integers, $k \geq 2$, $1 \leq s \leq k-1$ with $\gcd(s+2, k) = 1 = \gcd(s, k)$ and $\lambda \in K \setminus \{0\}$, we have $\text{HH}^2(\Lambda) \neq 0$.

Proof. Consider the element $h_1 + \text{Im } d_2$ of $\text{HH}^2(\Lambda)$ where h_1 is given as in Proposition 3.5 by

$$\begin{aligned} \mathfrak{o}(f_{1,1}^2) \otimes \mathfrak{t}(f_{1,1}^2) = e_1 \otimes e_{s+1} &\mapsto \rho_1, \\ \text{else} &\mapsto 0. \end{aligned}$$

Suppose for contradiction that $h_1 \in \text{Im } d_2$. Then $h_1(e_1 \otimes e_{s+1}) = f A_2(e_1 \otimes e_{s+1})$. So $\rho_1 = c'_1 \rho_1$ and so $c'_1 = 1$. Also $h_1(e_i \otimes e_{s+i}) = f A_2(e_i \otimes e_{s+i})$ where $i \in \{2, \dots, k\}$. Then $0 = c'_i \rho_i$, where $i \in \{2, \dots, k\}$. But this contradicts having $\sum_{i=1}^k c'_i = 0$. Therefore $h_1 \notin \text{Im } d_2$, that is, $h_1 + \text{Im } d_2 \neq 0 + \text{Im } d_2$. So $h_1 + \text{Im } d_2$ is a non-zero element in $\text{HH}^2(\Lambda)$. \square

Note that we can also define maps $h_i : Q^2 \rightarrow \Lambda$ by

$$\begin{aligned} \mathfrak{o}(f_{1,i}^2) \otimes \mathfrak{t}(f_{1,i}^2) &\mapsto \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p} = \rho_i, \\ \text{else} &\mapsto 0. \end{aligned}$$

for $i = 2, \dots, k$. However, h_1, h_2, \dots, h_k all represent the same element $h_1 + \text{Im } d_2$ of $\text{HH}^2(\Lambda)$.

As we have found a non-zero element in $\text{HH}^2(\Lambda)$ we know that $\dim \text{HH}^2(\Lambda) \geq 1$. In the case $1 \leq s \leq k-2$ we have the following result, the proof of which is immediate from Proposition 3.2, Corollary 3.4 and Theorem 3.6.

Proposition 3.7. For $\Lambda = \Lambda(p, q, k, s, \lambda)$ where $1 \leq s \leq k-2$, we have $\dim \text{Ker } d_3 = k$ and $\dim \text{HH}^2(\Lambda) = 1$.

For the case $s = k - 1$, we need more details to find $\text{Ker } d_3$. Following [4] we may choose the set f^3 to consist of the following elements:

$$\{f_{1,i}^3, f_{2,i}^3, f_{3,i,t'}^3, f_{4,i,j}^3\}, \text{ where}$$

$$\begin{aligned} f_{1,i}^3 &= f_{1,i}^2 \alpha_{(i-1,0)} \alpha_{(i-1,1)} + \beta_{(i+1)^0} \beta_{(i+1)^1} \cdots \beta_{(i+1)^p} \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q-1)} f_{3,i}^2 \alpha_{(i-1,1)} \\ &= \alpha_{(i,0)} f_{5,i,1}^2 \\ &\quad \in e_i K \mathcal{Q} e_{(i-1,2)} \text{ where } i \in \{2, \dots, k\}, \\ f_{1,1}^3 &= f_{1,1}^2 \alpha_{(k,0)} \alpha_{(k,1)} + \lambda \beta_{2^0} \beta_{2^1} \cdots \beta_{2^p} \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q-1)} f_{3,1}^2 \alpha_{(k,1)} \\ &= \alpha_{(1,0)} f_{5,1,1}^2 \\ &\quad \in e_1 K \mathcal{Q} e_{(k,2)}, \\ f_{2,i}^3 &= f_{1,i}^2 \beta_{i^0} \beta_{i^1} - \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^{(p-1)}} f_{2,i}^2 \beta_{i^1} \\ &= -\beta_{(i+1)^0} f_{4,i+1,1}^2 \\ &\quad \in e_i K \mathcal{Q} e_{i^2} \text{ where } i \in \{2, \dots, k\}, \\ f_{2,1}^3 &= f_{1,1}^2 \beta_{1^0} \beta_{1^1} - \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^{(p-1)}} f_{2,1}^2 \beta_{1^1} \\ &= -\lambda \beta_{2^0} f_{4,2,1}^2 \\ &\quad \in e_1 K \mathcal{Q} e_{1^2}, \\ f_{3,i,t'}^3 &= f_{5,i,t'}^2 \alpha_{(i-1,t'+1)} \\ &= \alpha_{(i,t')} f_{5,i,t'+1}^2 \\ &\quad \in e_{(i,t')} K \mathcal{Q} e_{(i-1,t'+2)} \text{ where } i \in \{1, \dots, k\} \text{ and } t' \in \{1, \dots, q-2\}, \\ f_{3,i,q-1}^3 &= f_{5,i,q-1}^2 \alpha_{(i-1,q)} - \alpha_{(i,q-1)} f_{3,i}^2 \alpha_{(i-1,1)} \cdots \alpha_{(i-1,q)} \beta_{(i-1)^0} \beta_{(i-1)^1} \cdots \beta_{(i-1)^p} \\ &= -\alpha_{(i,q-1)} \alpha_{(i,q)} f_{1,i-1}^2 \\ &\quad \in e_{(i,q-1)} K \mathcal{Q} e_{i-2} \text{ where } i \in \{1, 3, \dots, k\}, \\ f_{3,2,q-1}^3 &= \lambda f_{5,2,q-1}^2 \alpha_{(1,q)} - \alpha_{(2,q-1)} f_{3,2}^2 \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} \\ &= -\alpha_{(2,q-1)} \alpha_{(2,q)} f_{1,1}^2 \\ &\quad \in e_{(2,q-1)} K \mathcal{Q} e_k, \\ f_{3,i,q}^3 &= f_{3,i}^2 \alpha_{(i-1,1)} \cdots \alpha_{(i-1,q)} \beta_{(i-1)^0} \beta_{(i-1)^1} \cdots \beta_{(i-1)^p} \alpha_{(i-2,0)} - \alpha_{(i,q)} f_{1,i-1}^2 \alpha_{(i-2,0)} \\ &= \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i^p} \alpha_{(i-1,0)} \alpha_{(i-1,1)} \cdots \alpha_{(i-1,q-1)} f_{3,i-1}^2 \\ &\quad \in e_{(i,q)} K \mathcal{Q} e_{(i-2,1)} \text{ where } i \in \{1, 3, \dots, k\}, \\ f_{3,2,q}^3 &= f_{3,2}^2 \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^p} \alpha_{(k,0)} - \alpha_{(2,q)} f_{1,1}^2 \alpha_{(k,0)} \\ &= \lambda \alpha_{(2,q)} \beta_{2^0} \beta_{2^1} \cdots \beta_{2^p} \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q-1)} f_{3,1}^2 \\ &\quad \in e_{(2,q)} K \mathcal{Q} e_{(k,1)}, \\ f_{4,i,j}^3 &= f_{4,i,j}^2 \beta_{(i-1)^{(j+1)}} \\ &= \beta_{i^j} f_{4,i,j+1}^2 \\ &\quad \in e_{ij} K \mathcal{Q} e_{(i-1)^{(j+2)}} \text{ where } i \in \{1, \dots, k\} \text{ and } j \in \{1, \dots, p-2\}, \\ f_{4,i,p-1}^3 &= f_{4,i,p-1}^2 \beta_{(i-1)^p} - \beta_{i^{(p-1)}} f_{2,i}^2 \beta_{i^1} \cdots \beta_{i^p} \alpha_{(i-1,0)} \alpha_{(i-1,1)} \cdots \alpha_{(i-1,q)} \\ &= \beta_{i^{(p-1)}} \beta_{i^p} f_{1,i-1}^2 \\ &\quad \in e_{i^{(p-1)}} K \mathcal{Q} e_{i-2} \text{ where } i \in \{1, 3, \dots, k\}, \\ f_{4,2,p-1}^3 &= f_{4,2,p-1}^2 \beta_{1^p} - \lambda \beta_{2^{(p-1)}} f_{2,2}^2 \beta_{2^1} \cdots \beta_{2^p} \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \\ &= \beta_{2^{(p-1)}} \beta_{2^p} f_{1,1}^2 \\ &\quad \in e_{2^{(p-1)}} K \mathcal{Q} e_k, \\ f_{4,i,p}^3 &= f_{2,i}^2 \beta_{i^1} \cdots \beta_{i^p} \alpha_{(i-1,0)} \alpha_{(i-1,1)} \cdots \alpha_{(i-1,q)} \beta_{(i-1)^0} + \beta_{i^p} f_{1,i-1}^2 \beta_{(i-1)^0} \\ &= \beta_{i^p} \alpha_{(i-1,0)} \alpha_{(i-1,1)} \cdots \alpha_{(i-1,q)} \beta_{(i-1)^0} \beta_{(i-1)^1} \cdots \beta_{(i-1)^{(p-1)}} f_{2,i-1}^2 \\ &\quad \in e_{i^p} K \mathcal{Q} e_{(i-1)^1} \text{ where } i \in \{1, 3, \dots, k\}, \\ f_{4,2,p}^3 &= \lambda f_{2,2}^2 \beta_{2^1} \cdots \beta_{2^p} \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} + \beta_{2^p} f_{1,1}^2 \beta_{1^0} \\ &= \beta_{2^p} \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1} \cdots \beta_{1^{(p-1)}} f_{2,1}^2 \\ &\quad \in e_{2^p} K \mathcal{Q} e_{1^1}. \end{aligned}$$

$$\begin{aligned}
& \text{Thus the projective bimodule } Q^3 \text{ is } \bigoplus_{y \in f^3} \Lambda \mathfrak{o}(y) \otimes \mathfrak{t}(y) \Lambda \\
& = \bigoplus_{i=1}^k [(\Lambda e_i \otimes_{f_{1,i}^3} e_{(i-1,2)} \Lambda) \oplus (\Lambda e_i \otimes_{f_{2,i}^3} e_{i^2} \Lambda) \oplus \bigoplus_{t'=1}^{q-2} (\Lambda e_{(i,t')} \otimes_{f_{3,i,t'}^3} e_{(i-1,t'+2)} \Lambda) \\
& \quad \oplus (\Lambda e_{(i,q-1)} \otimes_{f_{3,i,q-1}^3} e_{i-2} \Lambda) \oplus (\Lambda e_{(i,q)} \otimes_{f_{3,i,q}^3} e_{(i-2,1)} \Lambda) \oplus \bigoplus_{j=1}^{p-2} (\Lambda e_{ij} \otimes_{f_{4,i,j}^3} e_{(i-1)(j+2)} \Lambda) \\
& \quad \oplus (\Lambda e_{i(p-1)} \otimes_{f_{4,i,p-1}^3} e_{i-2} \Lambda) \oplus (\Lambda e_{ip} \otimes_{f_{4,i,p}^3} e_{(i-1)^1} \Lambda)].
\end{aligned}$$

Now we determine $\text{Ker } d_3$ in the case $s = k - 1$. Let $h \in \text{Ker } d_3$, so $h \in \text{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Recall that for $i \in \{1, \dots, k\}$, h is given by

$$\begin{aligned}
\mathfrak{o}(f_{1,i}^2) \otimes \mathfrak{t}(f_{1,i}^2) & \mapsto d_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} + d'_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)}, \\
\mathfrak{o}(f_{3,i}^2) \otimes \mathfrak{t}(f_{3,i}^2) & \mapsto d''_i \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} \alpha_{(i-1,0)}, \\
& \text{else} \mapsto 0,
\end{aligned}$$

where d_i, d'_i, d''_i are in K .

Then for $i \in \{1, \dots, k\}$, we have $hA_3(e_i \otimes_{f_{1,i}^3} e_{(i-1,2)})$

$$\begin{aligned}
& = h(e_i \otimes_{f_{1,i}^2} e_{i-1}) \alpha_{(i-1,0)} \alpha_{(i-1,1)} \\
& \quad + \beta_{(i+1)^0} \beta_{(i+1)^1} \cdots \beta_{(i+1)^p} \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q-1)} h(e_{(i,q)} \otimes_{f_{3,i}^2} \alpha_{(i-1,1)}) \\
& \quad - \alpha_{(i,0)} h(e_{(i,1)} \otimes_{f_{5,i,1}^2} e_{(i-1,2)}) \\
& = d_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} \alpha_{(i-1,0)} \alpha_{(i-1,1)} \\
& \quad + d'_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \alpha_{(i-1,0)} \alpha_{(i-1,1)} \\
& \quad + d''_i \beta_{(i+1)^0} \beta_{(i+1)^1} \cdots \beta_{(i+1)^p} \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} \alpha_{(i-1,0)} \\
& = 0.
\end{aligned}$$

In a similar way we can show that $hA_3(e_1 \otimes_{f_{1,1}^3} e_{(k,2)}) = 0$.

For $i \in \{2, \dots, k\}$, we have $hA_3(e_i \otimes_{f_{2,i}^3} e_{i^2})$

$$\begin{aligned}
& = h(e_i \otimes_{f_{1,i}^2} e_{i-1}) \beta_{i^0} \beta_{i^1} \\
& \quad - \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{i(p-1)} h(e_{ip} \otimes_{f_{2,i}^2} e_{i^1}) \beta_{i^1} \\
& \quad + \beta_{(i+1)^0} h(e_{(i+1)^1} \otimes_{f_{4,i+1,1}^2} e_{i^2}) \\
& = d_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} \beta_{i^0} \beta_{i^1} + d'_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \\
& = d'_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1}.
\end{aligned}$$

As $h \in \text{Ker } d_3$ we have $d'_i = 0$ for $i \in \{2, \dots, k\}$.

Similarly it can be shown that $hA_3(e_1 \otimes_{f_{2,1}^3} e_{1^2}) = d'_1 \alpha_{(1,0)} \alpha_{(1,1)} \cdots \alpha_{(1,q)} \beta_{1^0} \beta_{1^1}$ so that $d'_1 = 0$.

We also have $hA_3(\mathfrak{o}(f_{3,i,t'}^3) \otimes_{f_{3,i,t'}^3} \mathfrak{t}(f_{3,i,t'}^3)) = 0$ for $i \in \{1, \dots, k\}$ and $t' \in \{1, \dots, q\}$. Finally, putting $hA_3(\mathfrak{o}(f_{4,i,j}^3) \otimes_{f_{4,i,j}^3} \mathfrak{t}(f_{4,i,j}^3)) = 0$ does not give any new information for $i \in \{1, \dots, k\}, j \in \{1, \dots, p\}$.

Thus h is given by

$$\begin{aligned}
\mathfrak{o}(f_{1,i}^2) \otimes \mathfrak{t}(f_{1,i}^2) & \mapsto d_i \alpha_{(i,0)} \alpha_{(i,1)} \cdots \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} \text{ for } i \in \{1, 2, \dots, k\}, \\
\mathfrak{o}(f_{3,i}^2) \otimes \mathfrak{t}(f_{3,i}^2) & \mapsto d''_i \alpha_{(i,q)} \beta_{i^0} \beta_{i^1} \cdots \beta_{ip} \alpha_{(i-1,0)} \text{ for } i \in \{1, \dots, k\}, \\
& \text{else} \mapsto 0,
\end{aligned}$$

where d_i, d''_i for $i \in \{1, \dots, k\}$ are in K . It is clear that there is no dependency between d_i, d''_i , and therefore $\dim \text{Ker } d_3 = 2k$.

Proposition 3.8. *For $\Lambda = \Lambda(p, q, k, s, \lambda)$ and $s = k - 1$, we have $\dim \text{Ker } d_3 = 2k$.*

Using Propositions 3.2, 3.7, 3.8 and Theorem 3.6 we get the main result of this paper.

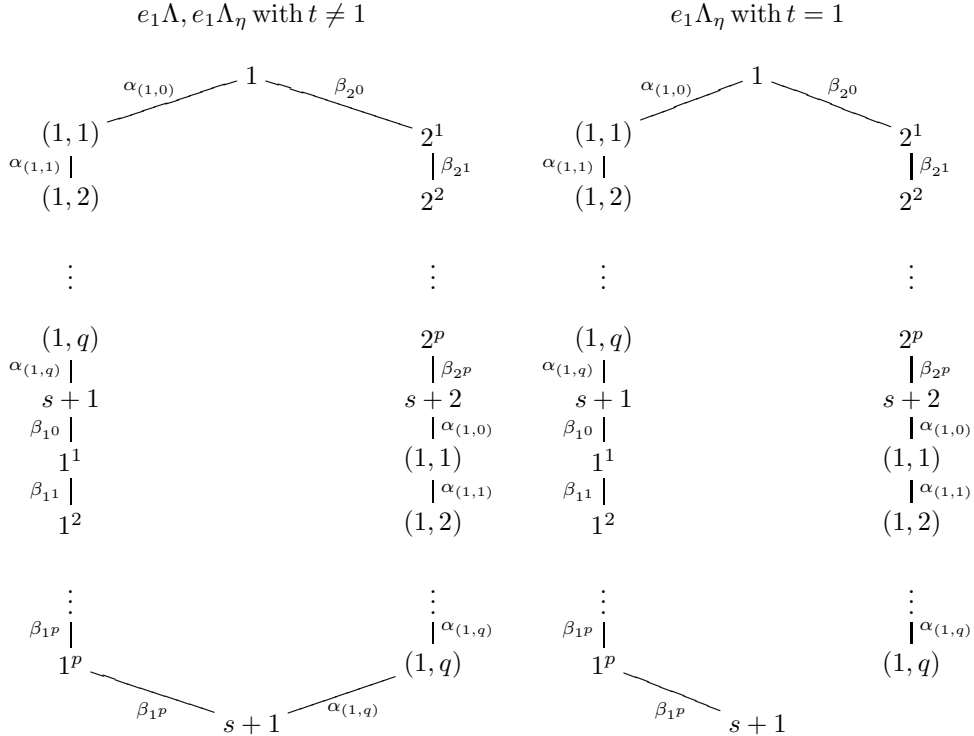
Theorem 3.9. For $\Lambda = \Lambda(p, q, k, s, \lambda)$ where p, q, s, k are integers such that $p, q \geq 0, k \geq 2, 1 \leq s \leq k-1, \gcd(s, k) = 1, \gcd(s+2, k) = 1$ and $\lambda \in K \setminus \{0\}$, we have $\dim \mathrm{HH}^2(\Lambda) = 1$.

We conclude this paper by giving a deformation of Λ which arises from the non-zero element $h_1 + \mathrm{Im} d_2$ in $\mathrm{HH}^2(\Lambda)$.

Let $\eta = h_1 + \mathrm{Im} d_2$. Recall that $\rho_1 = \alpha_{(1,0)}\alpha_{(1,1)} \cdots \alpha_{(1,q)}\beta_{1^0}\beta_{1^1} \cdots \beta_{1^p}$. We introduce a new parameter t and define the algebra Λ_η to be the algebra $K\mathcal{Q}/I_\eta$ where I_η is the ideal generated by the following elements:

- (1) $f_{1,1}^2 - t\rho_1, f_{1,j}^2$ where $j \in \{2, \dots, k\}$,
- (2) for all $i \in \{1, \dots, k\}$, $f_{2,i}^2, f_{3,i}^2, f_{4,i,j}^2, f_{5,i,t'}^2$, where $j \in \{1, \dots, p-1\}$, $t' \in \{1, \dots, q-1\}$,
- (3) $\rho_1 a$ for all arrows a with $\mathbf{t}(\rho_1) = \mathbf{o}(a)$,
- (4) $a\rho_1$ for all arrows a with $\mathbf{t}(a) = \mathbf{o}(\rho_1)$.

We now need to show that $\dim \Lambda_\eta = \dim \Lambda$ to verify that Λ_η is indeed a deformation of Λ . First of all, it is clear that $\dim e_j \Lambda_\eta = \dim e_j \Lambda$ for all t and for all vertices e_j with $e_j \neq e_1$. Now we consider $e_1 \Lambda$ and $e_1 \Lambda_\eta$ with $t \neq 1$, and $e_1 \Lambda_\eta$ with $t = 1$. These projective modules are described as follows:



In each case we see that $\dim e_1 \Lambda = \dim e_1 \Lambda_\eta = 2p + 2q + 4$ for all t . Hence $\dim \Lambda_\eta = \dim \Lambda$. Moreover, when $t = 1$ the algebras Λ and Λ_η are not isomorphic since, in this case, Λ_η is not self-injective. Thus we have found a non-trivial deformation of Λ .

Theorem 3.10. *With Λ, η , and Λ_η as defined above, then Λ_η is a non-trivial deformation of Λ . Moreover, the algebras Λ and Λ_η are socle equivalent.*

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